Probability Theory
2023/24 Period IIb
Instructor: Gilles Bonnet
Re-Exam
8/7/2024
Duration: 2 hours

Solutions

Question:	1	2	3	4	5	6	Total
Points:	10	10	15	16	16	23	90
Score:							

1. (10 points) How many different "words" can one form with all the letters of the word "BANANA"?

   Remark 1: Any sequence of letters is considered a word, even if it does not form a meaningful word in English.

   Remark 2: Your solution should include a short explanation of your reasoning, or the computation should be written in a way which makes it obvious to the corrector what is the reasoning, or both.

   Remark 3: In order to get full points, your final answer should be in a form of an integer number (not a formula).

   Solution: There are $\binom{6}{1}$ ways to choose the position of B.
   Once B is fixed, it remains 5 positions available for the three A's, thus we have $\binom{5}{3}$ ways to choose these positions.
   Then it remains two positions where we put the two N's.
   Therefore, the total number of different words is

   $$\binom{6}{1}\binom{5}{3}\binom{2}{2} = 6 \cdot \frac{5 \times 4}{2} \cdot 1 = 60.$$

2. (10 points) Let $m, n \in \mathbb{N}$. Let X be a continuous random variable with density function

   $$f_X(x) = \begin{cases} nx^{n-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Let $Y = X^{1/m}$. Find the density function of Y.

Solution: There are (at least) two distinct approaches to this problem. Both equally valid, the correct completion of any of them is sufficient to get full points.:

Approach 1: *Using the cumulative distribution function (CDF) method.*

For $y > 0$:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) \\ &= \mathbb{P}(X^{1/m} \le y) \\ &= \mathbb{P}(0 \le X \le y^m) && \text{since } X \text{ is positive} \\ &= \int_0^{y^m} f_X(x)\,\mathrm{d}x \\ &= \int_0^{y^m} nx^{n-1}\,\mathrm{d}x \\ &= [x^n]_0^{y^m} \\ &= y^{mn}. \end{aligned}$$

Therefore, the density function of Y is

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = mny^{mn-1}$$

if $y \in [0,1]$ and $f_Y(y) = 0$ otherwise.

Approach 2: *Using the transformation method.*

Set $g : [0,1] \to [0,1]$ such that $g(x) = x^{1/m}$. Then $g^{-1}(y) = y^m$. Therefore (by theorem 4.4.2. in the lecture notes), for we have

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| \\ &= n(y^m)^{n-1} m y^{m-1} \\ &= mny^{mn-1}, \end{aligned}$$

if $y \in [0,1]$ and $f_Y(y) = 0$ otherwise.

3. (15 points) Let $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$ be independent Poisson random variables. What is the distribution of $X_1 + X_2$? Justify your answer by writing down the appropriate computation.

Solution: For any integer $k \geq 0$,

$$\begin{aligned} f_{X_1+X_2}(k) &= \sum_{x_1=0}^{k} f_{X_1}(x_1) \cdot f_{X_2}(k-x_1) \\ &= \sum_{x_1=0}^{k} \frac{(\lambda_1)^{x_1}}{(x_1)!} \cdot e^{-\lambda_1} \cdot \frac{(\lambda_2)^{k-x_1}}{(k-x_1)!} \cdot e^{-\lambda_2} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{x_1=0}^{k} \frac{k!}{(x_1)!(k-x_1)!} (\lambda_1)^{x_1} (\lambda_2)^{k-x_1} \\ &= \frac{(\lambda_1+\lambda_2)^k}{k!} \cdot e^{-(\lambda_1+\lambda_2)}. \end{aligned}$$

Thus $\boxed{X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)}$.

Note that the above is Theorem 10.1.3 and its proof in the lecture note.

4. A student takes an exam with $n = 192$ questions. They get the answer to each question correctly with probability $p = \frac{3}{4}$, independently of all others. In order to pass the student needs to answer correctly at least 122 questions.
Let X_i be the indicator variable of the event that the student answers correctly the i-th question.
Let X be the number of correct answers.
Remark: $192 = 2^6 \times 3$.

   (a) (3 points) Compute the expected value of X.

   Solution: X is a binomial random variable with parameters $n = 192$ and $p = 3/4$. Therefore

   $$\mathbb{E}X = np = 192 \times \frac{3}{4} = 2^4 \times 3^2 = 144.$$

   (b) (3 points) Compute the variance of X.

   Solution: X is a binomial random variable with parameters $n = 192$ and $p = 3/4$. Therefore

   $$\operatorname{Var} X = np(1-p) = 192 \times \frac{3}{4} \times \frac{1}{4} = 2^2 \times 3^2 = 6^2 = 36.$$

   (c) (10 points) Compute an approximation of the probability that the student passes.

Solution: Note that $X = X_1 + \ldots + X_n$. Thus

$$\begin{aligned}\mathbb{P}(\text{pass}) &= \mathbb{P}(X \geq 122) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq 122\right) \\ &= \mathbb{P}\left(\frac{\left(\sum_{i=1}^{n} X_i\right) - \mathbb{E}\left(\sum_{i=1}^{n} X_i\right)}{\sqrt{\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)}} \geq \frac{122-144}{6}\right).\end{aligned}$$

Thus, the central limit theorem gives

$$\begin{aligned}\mathbb{P}(\text{pass}) &\approx \mathbb{P}(Z \geq \frac{122-144}{6}) \\ &= \mathbb{P}(Z \geq -3.666...) \\ &\simeq \mathbb{P}(Z \leq 3.67...) \\ &= 0,99988...\end{aligned}$$

where Z is a standard normal random variable.
ADDITIONAL REMARK: Since X is integer, $\mathbb{P}(X \geq 122) = \mathbb{P}(X > 121)$ which can lead to the following approximation:

$$\begin{aligned}\mathbb{P}(\text{pass}) = \mathbb{P}(X > 121) \approx \mathbb{P}(Z > \frac{121-144}{6}) = \mathbb{P}(Z > -\frac{23}{6}) = \mathbb{P}(Z < \frac{23}{6}) \\ \approx \mathbb{P}(Z < 3.83) = 0.99994...\end{aligned}$$

Therefore, any answer between 0.99988 and 0.99994 is acceptable.

5. (a) (8 points) State and prove Markov's inequality.

Solution: (This is Theorem 12.2.2 in the lecture notes)
Let $a > 0$ and Y be any non-negative random variable. Then,

$$\mathbb{P}(Y \geq a) \leq \frac{1}{a}\mathbb{E}[Y].$$

Proof:

$$\mathbb{P}(Y \geq a) = \mathbb{E}\left[\mathbf{1}\left\{Y \geq a\right\}\right] \leq \mathbb{E}\left[\frac{Y}{a}\right] = \frac{1}{a}\mathbb{E}[Y].$$

(b) (8 points) State and prove Chebyshev's inequality.

Solution: (This is Theorem 12.2.3 in the lecture notes)
If X is any random variable and $a > 0$ then

$$\mathbb{P}\left(|X - \mathbb{E}X| \geq a\right) \leq \frac{\operatorname{Var}(X)}{a^2}.$$

Proof: The random variable $Z := (X - \mathbb{E}X)^2$ is nonnegative. Hence we can apply Markov's inequality:

$$\mathbb{P}\left(|X - \mathbb{E}X| \geq a\right) = \mathbb{P}((X - \mathbb{E}X)^2 \geq a^2) \overset{\text{Markov}}{\leq} \frac{\mathbb{E}Z}{a^2} = \frac{\mathrm{Var}(X)}{a^2},$$

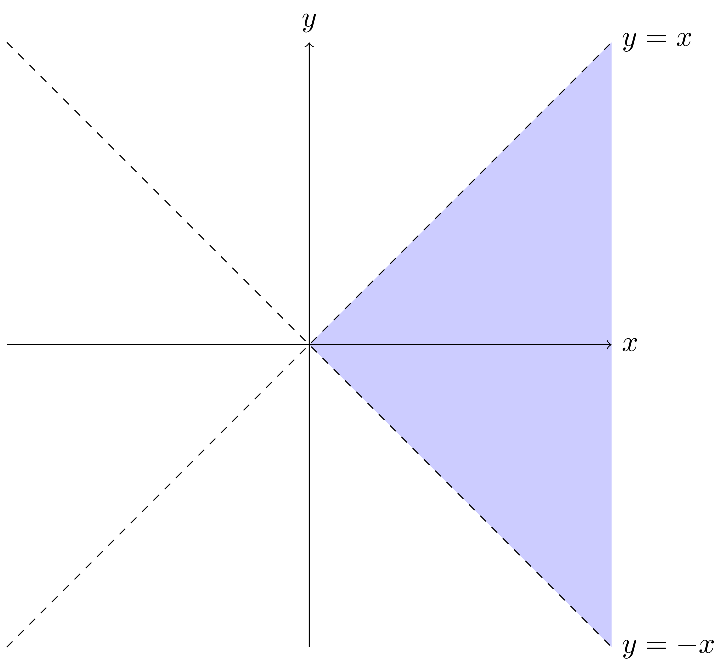
where we use the definition of $\mathrm{Var}(X)$ for the last equation.

6. The joint density of X and Y is given by

$$f(x, y) = C(x - y)e^{-x}\mathbf{1}(-x < y < x).$$

(a) (4 points) Find the value of the constant C.
Hint: You can use that $\int_0^\infty e^{-x}x^n \,\mathrm{d}x = n!$ for any $n \in \mathbb{N}$.

Solution: Drawing the domain $\{(x, y) : -x < y < x\}$ is not necessary but helps a lot:



From

$$1 = \int f(x, y)\,\mathrm{d}y\,\mathrm{d}x = C\int_0^\infty e^{-x}\int_{-x}^{x}(x - y)\,\mathrm{d}y\,\mathrm{d}x = C\int_0^\infty e^{-x}2x^2\,\mathrm{d}x = 4C,$$

one gets $\boxed{C = 1/4}$

(b) (9 points) Find the density function f_Y of Y.

Solution: For $y > 0$:

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \\ &= \int_y^\infty C(x-y)e^{-x}\,\mathrm{d}x \\ &= C\int_0^\infty ue^{-(y+u)}\,\mathrm{d}u \\ &= Ce^{-y}\int_0^\infty ue^{-u}\,\mathrm{d}u \\ &= Ce^{-y}\cdot(1!) \\ &= \boxed{\frac{1}{4}e^{-y}}. \end{aligned}$$

For $y < 0$:

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \\ &= \int_{-y}^\infty C(x-y)e^{-x}\,\mathrm{d}x \\ &= C\left[-xe^{-x} - e^{-x} + ye^{-x}\right]_{-y}^\infty \\ &= \boxed{\frac{1}{4}(-2ye^y + e^y)}. \end{aligned}$$

(c) (5 points) Find $\mathbb{E}[Y]$.

Solution:

$$\begin{aligned} \mathbb{E}[Y] &= \int_{\mathbb{R}} y f_Y(y)\,\mathrm{d}y \\ &= \frac{1}{4}\left[\int_{-\infty}^0 (-2y^2e^y + ye^y)\,\mathrm{d}y + \int_0^\infty ye^{-y}\,\mathrm{d}y\right] \\ &= \frac{1}{4}\left[-\int_0^\infty (2u^2e^{-u} + ue^{-u})\,\mathrm{d}u + 1\right] \\ &= \frac{1}{4}\left[-(2\times(2!) + 1!) + 1\right] \\ &= \boxed{-1}. \end{aligned}$$

(d) (2 points) Let f_X denote the density function of X. Show that $f_X(x) > 0$ for any $x > 0$.

Solution: For $x > 0$, $f_X(x) = \int_{-x}^{x} C(x-y)e^{-x}dy = Ce^{-x}\int_0^{2x} t dt > 0$.

(e) (3 points) Determine if X and Y are independent.

Solution: They are $\boxed{\text{not independent}}$, because $f \neq f_X \times f_Y$ as we can see with $(x, y) = (1, 2)$ for example. Indeed $f(1, 2) = 0 \neq f_X(1) \times f_Y(2)$ since $f_X(1) = \int_{-1}^{1} C(1-x)e^{-1}dx > 0$ and $f_Y(2) = \frac{1}{4}e^{-2}$.